Mid-Semester Exam — ISI Bangalore — March 2018

Introduction to Algebraic Geometry — 100 Points

Notes.

(a) Justify all your steps. You may use any result proved in class unless you have been asked to prove the same.

(b) $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers.

(c) By default, k denotes an algebraically closed field and $\mathbb{A}^n_k$ is the affine n-space over k. By default, the polynomial ring of functions on $\mathbb{A}^n_k$ is denoted as $k[X_1, \dots, X_n]$ while for $n = 1, 2, 3$ we also use the usual notation of X, Y, Z for the variables.

(d) We will use $\mathcal{V}(-)$ to denote the common zero locus (in suitable affine space) of any collection of polynomials and $\mathcal{I}(-)$ the ideal of functions vanishing on a given subset of affine space.

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1. [12 points] Let $f\colon \mathbb{A}^1_k \to \mathbb{A}^2_k$ be the polynomial map given by $t \mapsto (t^2, t^3)$. Describe the induced map of coordinate rings and describe its kernel.

2. [12 points] Prove that an affine algebraic set $Z \subset \mathbb{A}^n_k$ is irreducible if and only if $\mathcal{I}(Z)$ is a prime ideal in $k[X_1, \dots, X_n]$.

3. [12 points] Define what it means for a topological space to be noetherian. Prove that every subspace of a neotherian topological space is (quasi)-compact.

4. [12 points] Let $V = \mathcal{V}(XZ, YZ) \subset \mathbb{A}^3_k$. Find the irreducible and connected components of V and of $V \setminus \{(0, 0, 0)\}$.

5. [12 points] Let W_1, W_2 be disjoint nonempty closed subsets of an affine algebraic set Z. Prove that there exists a regular function f on Z such that f takes the value 1 everywhere on W_1 and 0 everywhere on W_2.

6. [12 points] Let $I \subset k[X_1, \dots, X_n]$ be an ideal such that $\mathcal{V}(I) = (0, \dots, 0)$. Let $\mathfrak{m} = (X_1, \dots, X_n)$. Prove that $\mathfrak{m}^r \subset I \subset \mathfrak{m}$ for some suitable integer r.

7. [14 points] Let A be a finitely generated k-algebra. Let $f \in A$. Construct a natural bijection between the set $\mathcal{M}$ of maximal ideals in A not containing f and the set $\mathcal{M}'$ of maximal ideals in A_f.

8. [14 points] Let V be the complement of finitely many points in $\mathbb{A}^3_k$. Compute the ring of regular functions $\mathcal{O}[V]$.